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Pursuit and evasion in non-convex domains of arbitrary dimension

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***Abstract*— Most results in pursuit-evasion games apply only to planar domains or perhaps to higher-dimensional domains which must be convex. We introduce a very general set of techniques to generalize and extend certain results on simple pursuit to non-convex domains of arbitrary dimension which satisfy a coarse curvature condition (the CAT(0) condition).**

I. Pursuit / Evasion

There is a significant literature on *pursuit-evasion* games, with natural motivations coming from robotics [10], [14], [24]. Such games involve one or more *evaders* in a fixed domain being hunted by one or more *pursuers* who win the game if the appropriate capture criteria are satisfied. Such criteria may be physical capture (the pursuers move to where the evaders are located) [12], [13], [20] or visual capture (there is a line-of-sight between a pursuer and an evader) [10], [23]. The types of pursuit games are many and varied: continuous or discrete time, bounded or unbounded speed, and constraints on admissible acceleration, energy expenditure, strategy, and sensing. For a quick introduction to the literature on pursuit games, see, e.g., [15], [10].

This paper focuses on one particular variable in pursuit games: the geometry and topology of the domain on which the game is played. The vast majority of the known results on pursuit-evasion are dependent on having domains which are two-dimensional or, if higher-dimensional, then convex. There has of late been a limited number of results for pursuit games on surfaces of revolution [11], cones [18], and round spheres [16]. Our results are complementary to these, in the sense that we work with domains having dimension higher than two, without constraints on being smooth or a manifold.

The principal contribution of this work is a significant extension of known results on convex or planar domains to domains of arbitrary dimension which satisfy a type of curvature constraint known as the CAT(0) condition. Roughly speaking, the CAT(0) condition is a measure of what triangles in a metric space (X, d) look like, and, in particular, how a triangle compares to a Euclidean triangle with the same three side lengths. A simple mnemonic for a CAT(0) space is that it is a metric space, all of whose geodesic triangles have an angle sum no greater than π. Examples of CAT(0) domains are numerous and include the following:

1) convex Euclidean domains;
2) simply-connected subsets of $\mathbb{E}^2$;
3) simply-connected Riemannian manifolds with nonpositive sectional curvature;
4) smooth Euclidean domains with boundaries having no more than one non-convex direction at each point;
5) simply-connected piecewise-Euclidean cubical complexes with no positive discrete curvature at the vertices;
6) Euclidean rectangular prisms with certain cylindrical sets removed;
7) simply-connected unions of convex sets which have no triple intersections.

Our goal in this paper is to motivate the adoption of CAT(0) techniques in this and other areas of robotics in which the generalization of results from 2-d to higher dimensions is problematic. Decades of work by geometers in CAT(0) and more general Alexandrov geometry (geometry of spaces of bounded curvature) forms a powerful set of tools which are not very visible outside of mathematics departments (see [4], [5] and §III below.). The proofs of pursuit/evasion results in this paper are all very simple and very short, given the appropriate standard results from comparison geometry. We extend results to CAT(0) spaces in a dimension-independent manner, and, often, examples which are of high dimension are no more difficult than those with dimension two: the same proofs cover all cases.

After giving a motivational example of a simple pursuit problem in the plane (§II-B), we motivate the notion of comparison triangles, total curvature bounds, and their utility in simple pursuit problems. We then present a brief primer on CAT(0) geometry in §III, followed by a more technical result on growth rates of total curvature in §IV. These tools are used in §V to solve problems involving simple pursuit curves. We conclude this note with results on escape criteria (§VI), contrasts with the positive curvature case (§VII), and a series of remarks and open directions (§VIII).

II. A QUICK SUMMARY OF PURSUIT

A. Definitions

In this paper, we focus on pursuit-evasion problems involving *capture*, in which the pursuer wins if the distance to the evader limits to within some fixed threshold. We do not at this time consider *visibility games* for which the capture criterion is a line-of-sight between pursuer and evader, though we believe that our techniques are applicable to such. Time can be discrete or continuous: we will present proofs in both cases as examples of CAT(0) techniques. All pursuit in this paper assumes unit speed: generalizations to equal non-unit speeds is immediate.

The paper [15] gives an excellent introduction to various types of pursuit and evasion, with a particular emphasis on the allowable types of communication and coordination between multiple pursuers. A pursuit algorithm is said to be

- **successful** if the pursuer gets sufficiently close to the evader in finite time, independent of the evader's strategy;
- **local** if the pursuers are not permitted to exchange information after the initial time step;
- **oblivious** if the pursuers never exchange any information;
- **memoryless** if the algorithm depends solely on the instantaneous data of all pursuer and evader locations.
- **simple** if the algorithm has the pursuer travel at maximal speed toward the instantaneous position of the evader.

The primary contributions of this paper are as follows. (1) On any compact CAT(0) domain, we show that simple pursuit is always successful; (2) On compact domains which are not CAT(0), we observe that simple pursuit is not always successful; and (3) On an arbitrary CAT(0) domain, we give a necessary evasion criterion in terms of *total curvature* of a putative escape path with respect to simple pursuit.

With the possible exception of (3), none of these results is surprising when applied to the Euclidean plane. The principal contribution of this paper is the extension of these results to higher-dimensional non-convex CAT(0) domains and the development of total-curvature techniques for analyzing such problems.

It may be argued that pursuit problems are not physically relevant on domains of dimension higher than two. We disagree. Pursuit in three-dimensional domains finds some justification in the fact that the physical world has more than two spatial dimensions. Better still, as robotics researchers well know, configuration spaces of realistic robotic systems are rarely low-dimensional. It is not hard to imagine the following scenario. Consider a robot arm (or similar system) which a user controls by trying to "chase" a moving goal configuration. To make it interesting, assume the controller has no knowledge of the future goal states. Does the algorithm of "simple pursuit" (move towards the goal in C-space) always eventually converge to the moving goal configuration? The present paper answers this question when the configuration space is CAT(0).

B. A motivational example

We begin with a simple pursuit problem in a convex Euclidean $\mathcal{D}$, and use this to motivate certain core ideas in comparison geometry. Assume that there is a single pursuer and a single evader starting at locations P_0 and E_0 respectively. The evader moves from E_t to E_{t+1}, a point at distance 1. The pursuer moves to P_{t+1}, the point along a straight line from P_t to E_t of distance 1 from P_t. This is well-defined since $\mathcal{D}$ is convex. For discrete-time simple pursuit, we always assume $d(P_0, E_0) > 1$; given P_0 and the sequence $\{E_t\}$, we say that P wins if for every $C > 1$, $\|P_t - E_t\| < C$ for some t; otherwise, E wins.

The following result is well-known and easy to prove.

Theorem 1: Discrete-time simple pursuit on any compact convex Euclidean domain $\mathcal{D}$ is always successful.

We sketch a proof which will generalize to certain non-convex domains. Consider the situation illustrated in Fig. 1. The three points, P_{t+1}, E_t, and E_{t+1} form a triangle with side lengths $L_t - 1$, 1 and L_{t+1}, where $L_t = \|P_t - E_t\|$ for each $t \in \mathbb{N}$. The triangle inequality states that any side is no longer than the sum of the other two: thus, $L_{t+1} \leq (L_t - 1) + 1 = L_t$. By this monotonicity, $\lim_{t\to\infty} L_t$ exists, and both sides of our triangle inequality have the same limit C. Suppose E wins. Then $C > 1$, from which it follows that the angle $\alpha_t = \angle E_t P_{t+1} E_{t+1}$ approaches zero, and both E and P travel on an almost-straight line segment in $\mathcal{D}$. In particular, the distance $\|E_s - E_{s+t}\|$ grows linearly in t for s sufficiently large. This contradicts the compactness of $\mathcal{D}$ and implies that the pursuer always wins.

There are two ingredients of this proof that generalize to certain nonconvex domains. The first is the use of the Euclidean geometry of the triangle of Fig. 1 to

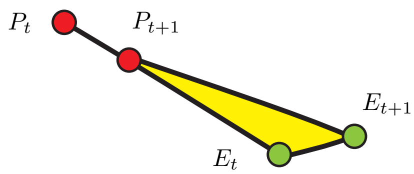

Fig. 1. Discrete time simple pursuit in a convex Euclidean domain.

show that $L_{t+1} \leq L_t$. The second is the argument that

$$\lim_{t\to\infty} L_t > 1 \text{ and } \lim_{t\to\infty} \alpha_t = 0,$$

imply that $\mathcal{D}$ contains an arbitrarily long line segment.

III. A CAT(0) PRIMER

A. Definitions

This subsection serves as the briefest possible summary of the basic definitions behind CAT(0) geometry. The idea of a CAT(0) space begins with a complete metric space (X, d) which is a *geodesic space*, in the sense that between any two points there is a geodesic — a path in X for which path length agrees with metric distance in X. With this structure alone, one can discuss angles, curvatures, and other geometric features one normally associates with Riemannian geometry.

Triangles: Given a triple of points (p, q, r) in X, a *triangle* is a triple of geodesics pq, qr, and pr. Thus a “triangle” consists of three sides (but no ‘interior’).

Curvature: The key to discussing curvature for metric spaces is to examine triangles. Given a triangle $\triangle pqr$, construct a *comparison triangle* $\triangle\tilde{p}\tilde{q}\tilde{r}$ in the Euclidean plane $\mathbb{E}^2$ whose side lengths are the same as those in X. If X were itself a Euclidean space, then $\triangle pqr$ and $\triangle\tilde{p}\tilde{q}\tilde{r}$ would be indistinguishable: any two points of $\triangle pqr$ would have metric distance equal to the Euclidean distance of the comparable points of $\triangle\tilde{p}\tilde{q}\tilde{r}$. On the other hand, if X were a round 2-sphere — the model of positive curvature — then the geodesic triangle in X would be “fatter” than its Euclidean counterpart. Such girth is detectable by comparing the distance between points of $\triangle pqr$ in X with the comparison points of $\triangle\tilde{p}\tilde{q}\tilde{r}$. These *chord lengths* detect the presence of positive curvature (when chords in X are longer than those in $\mathbb{E}^2$) or of negative curvature (when chords in X are shorter than those in $\mathbb{E}^2$).

Angles: Fix a point $p \in X$ and a pair of geodesics, $\sigma_1(s)$ and $\sigma_2(t)$, which emanate from p. One can use comparison to $\mathbb{E}^2$, fixing a point $\tilde{p}$ and choosing for any s and t a comparison triangle $\triangle\tilde{\sigma_1}(s)\tilde{p}\tilde{\sigma_2}(t)$. The angle $\angle\sigma_1\sigma_2$ is defined to be

$$\angle\sigma_1\sigma_2 = \lim_{s,t\to 0} \angle\tilde{\sigma_1}(s)\tilde{p}\tilde{\sigma_2}(t), \quad (1)$$

if the limit exists.

CAT(0): The key definition for this paper is the following. A complete geodesic metric space (X, d) is CAT(0) if no triangle has a chord longer than that in its Euclidean comparison triangle. As explained above, this condition encodes nonpositive curvature. For such a space, there is a unique geodesic joining any two points, angles are always well-defined, and the angles of a triangle are no larger than their corresponding angles in the comparison triangle. This last fact yields the maxim that CAT(0) means *“No fat triangles,”* or, equivalently, that the angle sum is no greater than π.

This implies a number of global properties. For example, in any CAT(0) space, any closed curve can be shrunk to a point within the space: otherwise said, the space must be *simply-connected*. In addition, geodesics of a CAT(0) space between two points are unique and vary continuously with endpoints. See [4] for a very thorough introduction.

B. Examples

The following are a few concrete examples of CAT(0) spaces. Some of these are illustrated in Fig. 2.

Example 2: Any convex Euclidean domain is CAT(0), since any geodesic triangle in such a space lies within a Euclidean plane and thus is isometric to its planar comparison. Such spaces are everywhere flat.

Example 3: Consider the subset of $\mathbb{E}^2$ obtained by deleting the interior of a quadrant. The metric in this example is the *path metric*. In this space, triangles are the same as their planar counterparts, unless the origin happens to lie in the interior of the convex hull (in the full Euclidean plane) of the vertices. In this case, one or two edges of the geodesic triangle “bends” around the origin, yielding a triangle which is definitely “skinny.” This example can be thought of as having negative curvature concentrated at the origin.

Example 4: Any finite tree (a graph without cycles) outfitted with a metric is CAT(0). Here, any triangle in the tree has angle sum either π (if the vertices all lie on a single arc) or zero (if the vertices span a ‘Y’ in the tree). Such trees can be thought of as being flat on the edges and having negative curvature at the vertices with degree greater than two.

Example 5: A Euclidean domain with smooth boundary can be CAT(0) depending on how much the boundary bends away from the interior, according to a more general theorem in [1]. Specifically, a closed, simply-connected 3-dimensional Euclidean domain $\mathcal{D}$ with smooth boundary is CAT(0) if and only if the tangent plane at each boundary point p contains points arbitrarily close to p that are not in the interior of $\mathcal{D}$.

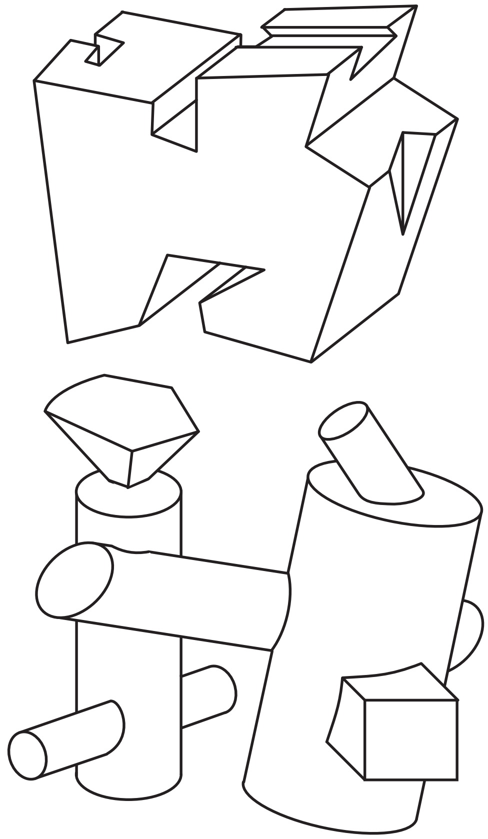

Fig. 2. Examples of solid 3-dimensional CAT(0) subsets of $\mathbb{E}^3$: [top] a cylindrically deleted cube; [bottom] a union of convex sets glued along convex subsets.

For example, a hyperboloid of one sheet is the intersection of two domains in $\mathbb{E}^3$ (the "internal" and the "external" pieces). The internal component is CAT(0). (For n-dimensional Euclidean domains, $n > 2$, the CAT(0) condition can be expressed by saying that at most one principal direction at each boundary point is tangent to a curve in the boundary which bends away from the interior.)

Example 6: Consider a space which is an N-dimensional cube $[0,1]^N$ with axis-aligned "cylindrical" sets drilled out. Specifically, consider a space of the form:

$$[0,1]^N - \bigcup_{i<j} \left\{ (x_k)_1^N : (x_i, x_j) \in \Delta_{i,j} \right\}, \qquad (2)$$

where for each $i < j$, $\Delta_{i,j}$ is the interior of a piecewise real-analytic subset of $[0,1] \times [0,1]$. Any such cylindrically deleted cube which is simply-connected is CAT(0): see Fig. 2[top] for an example.

Example 7: One can combine CAT(0) spaces to generate new examples. Cross products of CAT(0) spaces are clearly CAT(0) in the product metric. A more significant class of examples comes from gluing CAT(0) spaces along convex subsets. A subset A of a geodesic metric space is said to be *convex* if every geodesic between any two points in A is contained in A. It follows from a more general gluing theorem [4, Theorem II.11.1] that gluing two CAT(0) spaces along isometric convex subsets yields a CAT(0) space. For example, consider any simply-connected subset of $\mathbb{E}^n$ of the form

$$X = \bigcup_{\alpha} C_\alpha, \qquad (3)$$

where C_α is convex and there are no triple-intersections: $C_\alpha \cap C_\beta \cap C_\gamma = \emptyset$ for $\alpha \neq \beta \neq \gamma$. Then X is CAT(0). See Fig. 2[bottom].

C. Tools

Many of the basic concepts and tools available in Euclidean and Riemannian geometry apply naturally to CAT(0) spaces, even though these need be neither smooth nor manifolds. We use the following in later proofs.

1) The First Variation Theorem:: Let γ be a curve in a CAT(0) space. The *speed* of γ at $\gamma(0)$ is $\frac{d}{dt}\big|_{t=0} d(\gamma(0), \gamma(t))$. The following result is a generalization of the *First Variation Formula* from Riemannian geometry:

Theorem 8 ([5], [4]): Let X be a CAT(0) space, and γ_1 and γ_2 be unit speed curves in X parametrized by t. Let α_i, $i = 1, 2$ be the angle between γ_i and the geodesic connecting $\gamma_1(0)$ to $\gamma_2(0)$. Then

$$\frac{d}{dt}\bigg|_{t=0} d(\gamma_1(t), \gamma_2(t)) = -\cos\alpha_1 - \cos\alpha_2. \qquad (4)$$

2) Reshetnyak Majorization:: Reshetnyak proved a far-reaching generalization of the defining property of a CAT(0) space. It says in particular that any triangle can be filled in by the image of a distance-nonincreasing map from the inside of a model triangle, but more than that, there is no need to restrict attention to triangular closed curves. One can construct such a *majorizing map* for any closed curve:

Theorem 9 ([21]): Let γ be a closed curve in a CAT(0) space X. Then there is a closed curve $\widetilde{\gamma}$ which is the boundary of a convex region D in $\mathbb{E}^2$ and a distance-nonincreasing map $\varphi : D \to X$ such that the restriction of φ to $\widetilde{\gamma}$ is an arclength-preserving map onto γ.

Thus, any closed curve in X can be "filled in" by the image of a convex planar Euclidean set in a way that either preserves or reduces distances between points: see Fig. 3. This is a core idea in CAT(0) geometry: instead of worrying how a closed curve is situated in X, one pulls it back to $\mathbb{E}^2$ and works in the plane, knowing that distances, if distorted in this representation, are not increased.

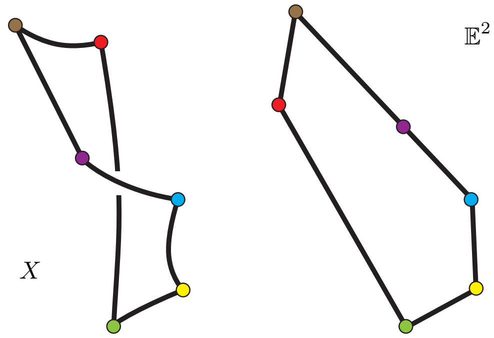

Fig. 3. Reshetnyak Majorization compares a closed curve in a CAT(0) space X [left] to the boundary of a convex Euclidean planar domain [right], with all "chords" in X being no longer than those of $\mathbb{E}^2$.

IV. Total curvature

The notion of a CAT(0) space generalizes the triangle arising in the simple pursuit problems on convex domains (Fig. 1). The second main ingredient of our extension of simple pursuit requires a new set of techniques which flow naturally from CAT(0) geometric principles.

A. Definition

In this section we work in a CAT(0) domain $\mathcal{D}$. The *total curvature* τ_σ of a piecewise-geodesic (or *polygonal*) curve σ is

$$\sum_j (\pi - \alpha_j), \tag{5}$$

where the $\alpha_j \geq 0$ are the angles at the interior vertices. It follows from CAT(0) triangle comparisons that if σ is inscribed in a polygonal curve γ, then $\tau_\sigma \leq \tau_\gamma$ [2]. Thus the total curvature τ_γ of any curve γ may be defined as the supremum of τ_σ over all polygonal σ inscribed in γ. Curves of finite total curvature in CAT(0) spaces are well-behaved, in the sense that they have unit-speed parametrizations, which have left and right unit velocity vectors at every point.

If $\mathcal{D}$ is a convex domain in $\mathbb{E}^n$, and γ is a curve with unit speed, then τ_γ equals the length of the curve γ'^+ of righthand unit tangent vectors in the unit sphere, with jump discontinuities replaced by great circular arcs [3]. In particular, if γ is smooth in $\mathbb{E}^2$, so that $\gamma'(t) = (\cos\theta(t), \sin\theta(t))$, then $\tau_\gamma = \int \kappa$, where $\kappa = |\gamma''| = |\theta'|$.

In following section, we show that conditions for capture or escape in a CAT(0) domain can be expressed in terms of the asymptotics of the *total curvature function* $\tau(t) := \tau_{\gamma|[0,t]}$, where γ is the path of the evader and t is its unit-speed parameter.

B. Sublinear total curvature and growth

The growth of a curve γ may be measured by its *circumradius function* c, where $c(t)$ is the circumradius up to time t: the smallest number $c(t)$ such that the path $\gamma|[0,t]$ lies in the ball of radius $c(t)$ about $\gamma(0)$. We show how restrictions on total curvature control this circumradius. The following theorem and its proof parallel a theorem of Dekster in the Riemannian setting [7], but uses Reshetnyak majorization to obtain a simple argument that moreover holds for any CAT(0) domain. The theorem is stated for the continuous case, but may be applied equally well to the discrete case ($t \in \mathbb{N}$, $d(\gamma(t), \gamma(t+1)) = 1$) by joining the vertices by geodesic segments of length 1 to obtain a polygonal curve.

Theorem 10: For any curve γ in a CAT(0) domain:

(a) If $\liminf_{t\to\infty} \tau(t)/t = 0$, then γ is unbounded.
(b) If $\tau(t)/t^a$ is bounded, for some $a \in (0,1)$, then $t^{1-a}/c(t)$ is bounded.

As an illustration, consider the spiral $\gamma(u) = (u\cos 2\pi u, u \sin 2\pi u)$ in $\mathbb{E}^2$. The total curvature function is linear in u, as is the circumradius, while the arclength t grows quadratically. This is the case $a = \frac{1}{2}$ in part (b), with $t/c(t)^2$ bounded.

Proof: Recall that $\liminf_{t\to\infty} \tau(t)/t$ is defined as $\lim_{t\to\infty}(inf_{u\geq t}\, \tau(u)/u)$. For part (a), we may suppose by approximation that any fixed initial segment $\gamma|[0,t]$ is polygonal. Subdivide $[0,t]$ into at most $\frac{\tau(t)}{\pi/2} + 1$ subintervals so that the restriction γ_i of γ to each subinterval has total curvature at most $\pi/2$. (If any angles are less than $\pi/2$, we first refine the polygon by cutting across each such angle with a short segment to obtain two angles of at least $\pi/2$.) Let ρ_i be the closed polygon consisting of γ_i and its chord σ_i. By Reshetnyak majorization, there is a closed convex curve $\widetilde{\rho}_i$ in $\mathbb{E}^2$ that majorizes ρ_i. Since a majorizing map preserves geodesics and does not increase angles, $\widetilde{\rho}_i$ is a closed polygon with the same sidelengths as ρ_i, consisting of a polygonal curve $\widetilde{\gamma}_i$ and its chord $\widetilde{\sigma}_i$, where the total curvature of $\widetilde{\gamma}_i$ is at most $\pi/2$.

Since $\widetilde{\gamma}_i$ is a convex curve in $\mathbb{E}^2$ having total curvature at most $\pi/2$, the ratio of its length to that of its chord is at most $\sqrt{2}$ (the ratio of two sides of an isosceles right triangle to its hypotenuse). Therefore

$$t \leq \left(\frac{\tau(t)}{\pi/2} + 1\right)\sqrt{2}\sup|\sigma_i|, \tag{6}$$

so

$$\frac{\tau(t)}{t} \geq \frac{\pi}{2}\left(\frac{1}{\sqrt{2}\sup|\sigma_i|} - \frac{1}{t}\right).$$

But if γ is bounded, so that $\sup|\sigma_i| < \infty$, it follows that $\tau(t)/t$ is bounded away from 0 for t sufficiently large. This proves Part (a).

For Part (b), if one substitutes $\tau(t) \leq At^a$ and $\sup|\sigma_i| \leq 2c(t)$ in (6), it is immediate that $t/c(t)^{1/(1-a)}$ is bounded. ■

V. Simple pursuit on CAT(0) domains

Having developed the appropriate tools, we generalize our motivational example to CAT(0) domains. We consider simple pursuit, in which the sole pursuer adopts the instantaneous strategy of moving toward the sole evader's current position, with the pursuer and evader moving at unit speed. Let P_0 and E_0 denote initial positions of the respective agents, where $d(P_0, E_0) > 1$. We break the problem into discrete-time and continuous-time versions for expository reasons, to illustrate the types of tools available in CAT(0) geometry.

A. Discrete time capture

Theorem 11: In the discrete time case, simple pursuit is successful on any compact CAT(0) domain $\mathcal{D}$.

Proof: Consider the situation illustrated in Fig. 4. Four points, P_t, E_t, E_{t+1} and P_{t+1}, form a degenerate geodesic quadrangle with side lengths L_t, 1, L_{t+1}, and 1, where $L_t = d(P_t, E_t)$ for each integer t. By drawing a comparison triangle in $\mathbb{E}^2$, one observes that $L_{t+1} \leq L_t$. By monotonicity, $\lim_{t\to\infty} L_t$ exists and the evader wins if and only if this limit is greater than 1. In this case, the angle α_t between the geodesics joining P_{t+1} to E_t and E_{t+1} vanishes in the limit, because the same is true for the comparison triangles. Hence, the total curvature of the P curve is sublinear and this curve is unbounded, contradicting the fact that the domain is compact. ⋄

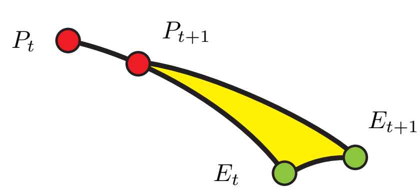

Fig. 4. A comparison triangle arising from a discrete time capture problem. This illustrates the naturality of the CAT(0) definition in the context of pursuit problems.

B. Continuous time capture

In this setting of simple pursuit, the pursuer and evader move along parameterized curves P_t and E_t (resp.) at constant unit speed. For each t, the velocity vector P'_t points along the geodesic from P_t to E_t. As follows from equation (7) below, L_t is nonincreasing, where $L_t = d(P_t, E_t)$. The evader wins if and only if $\lim_{t\to\infty} L_t = C > 0$.

Theorem 12: In the continuous time case, simple pursuit is successful on any compact CAT(0) domain $\mathcal{D}$.

Proof: Denote by α_t^P and α_t^E the angles between the (respective) velocity unit vectors P'_t and E'_t and the geodesic segment from E_t to P_t. According to the First Variation Theorem,

$$\frac{dL}{dt} = -\cos\alpha_t^E - \cos\alpha_t^P. \tag{7}$$

Since this is a pursuit curve, $\alpha_t^P = 0$. To avoid capture, it must be the case that $\alpha_t^E \to \pi$, meaning that the total curvature is sublinear and the pursuit curve is unbounded: contradiction. ⋄

VI. Escape

On a noncompact domain, the relevant question is whether the evader can escape when the pursuer adopts the simple pursuit-curve strategy, and, if so, what initial conditions lead to escape. Here we show that the pursuer still always wins if the circumradius of the evader does not grow fast enough, or, equivalently via Theorem 10, if the evader makes its path curve too much.

Theorem 13: On any CAT(0) domain $\mathcal{D}$, if the evader wins a simple pursuit game, then $\sqrt{t}/c(t)$ is bounded, where $c(t)$ is the evader's circumradius up to time t.

Proof: We present a proof for the case of discrete time. It suffices to show that $\tau(t)/\sqrt{t}$ is bounded, where $\tau(t)$ is the total curvature function of the pursuer, as is seen by taking $a = 1/2$ in Part (b) of Theorem 10.

Consider the configuration illustrated in Figure 4 with angle $\alpha_i = \angle E_i P_{i+1} E_{i+1}$ and comparison angle $\tilde{\alpha}_i$, where angle $\tilde{\alpha}_i \geq \alpha_i$. We start with the notation of Theorem 10, and assume again that the evader escapes, so $\lim L_t = C > 1$. Also $\lim \alpha_t = 0$ and the total curvature function of the pursuer is $\tau(t) = \sum_{i=1}^t \alpha_i$. We may assume that $\alpha_i \leq 1$ for all i. By the Law of Cosines applied to the comparison triangle,

$$1 = L_i^2 + (L_{i-1} - 1)^2 - 2L_i(L_{i-1} - 1)\cos\tilde{\alpha}_i. \tag{8}$$

Since $\alpha_i \leq 1$, Taylor's Theorem implies,

$$\cos\tilde{\alpha}_i \leq 1 - \frac{\tilde{\alpha}_i^2}{2} + \frac{\tilde{\alpha}_i^4}{24} \leq 1 - \frac{11}{24}\tilde{\alpha}_i^2. \tag{9}$$

It follows that

$$\begin{aligned} 0 \geq & L_i^2 + L_{i-1}^2 - 2L_{i-1} - 2L_iL_{i-1} + 2L_i + \\ & + \frac{11}{12}L_i(L_{i-1} - 1)\tilde{\alpha}_i^2, \end{aligned} \tag{10}$$

and therefore

$$\alpha_i^2 \le \tilde{\alpha}_i^2 \le \frac{12}{11}\frac{-(L_{i-1}-L_i)^2+2(L_{i-1}-L_i)}{L_i(L_{i-1}-1)} \le \frac{24}{11C(C-1)}(L_{i-1}-L_i). \tag{11}$$

Summing from $i = 1$ to t and using the Cauchy-Schwarz inequality (in the form $(v{\cdot}w)^2 \le (v{\cdot}v)(w{\cdot}w)$, where the entries of v are all 1's) gives the curvature bound:

$$\frac{\tau(t)^2}{t} = \frac{1}{t}\left(\sum_{i=1}^{t}\alpha_i\right)^2 \le \sum_{i=1}^{t}\alpha_i^2 \le \frac{24}{11C(C-1)}(L_0 - L_t) \le \frac{24(L_0-C)}{11C(C-1)}. \tag{12}$$

Thus $\tau(t)/\sqrt{t}$ is bounded as required. ⋄

VII. Domains with positive curvature

For non-convex domains in Euclidean n-space for $n \ge 3$, it is not necessarily the case that the pursuer always wins, even if the space is compact.

Example 14: Consider a flat annular strip $A = \{(r,\theta,z) \in \mathbb{E}^3 : r = 1, z \in [0,1]\}$. The evader wins by moving away from the pursuer along a geodesic circle of constant z in A (this being equivalent to running off to infinity in the locally isometric universal cover). The domain A is not simply-connected. However, we can attach a disc $\{z = 0, r \le 1\}$ to A and obtain a simply-connected (even contractible) space B on which evasion is always possible.

This perfectly highlights the topological and geometric obstructions to capture. The domain A is flat, but capture is prevented by topological reasons: the fundamental group is the obstruction. The domain B is contractible, but, even though it is built from flat pieces, there is positive curvature concentrated at the rim $\{z = 0, r = 1\}$. This example may easily be generalized to a 3-dimensional contractible domain in $\mathbb{E}^3$ by thickening B to a thin shell, the boundary of which possesses regions of positive curvature. Higher-dimensional examples are likewise easily constructed.

Positive curvature is not necessarily an obstruction to capture in the way that the fundamental group is.

Example 15: Consider the round 2-dimensional sphere $S^2 \subset \mathbb{E}^3$ and let H denote the upper hemisphere $H = \{(x,y,z) \in S^2 : z \ge 0\}$ with spherical coordinates (θ,ϕ). The standard argument for pursuit in a Euclidean disc works here to show that the pursuer can always win. Assume without a loss of generality that P_0 is at the north pole, $\phi = 0$. The pursuer can perfectly track the longitude θ of the evader at each step. The latitude ϕ of the pursuer is always less than that of the evader, but can be increased by an amount at least $\cos\phi$ times the jump size. In the limit, the latitude of the pursuer approaches that of the evader and capture occurs.

This domain H is of course not CAT(0); the round sphere is a model of positive curvature. Note, however, that the argument breaks down for a slightly larger portion of a sphere that dips below the equator. In this case, evasion may occur.

VIII. On CAT(0) geometry

We have demonstrated that CAT(0) geometry is efficacious in the context of pursuit-evasion games. In particular, we have demonstrated generalizations of known results in planar Euclidean convex domains to domains of arbitrary dimension which are not necessarily either smooth or locally Euclidean or convex. The proofs, though in a language which is perhaps unfamiliar to researchers in robotics or differential games, are essentially "two dimensional" proofs. The genius of the CAT(0) condition is that two-dimensional intuitions and techniques regulate all. The existing literature on pursuit-evasion games on non-planar and non-convex domains has as its focus the work of Melikyan [18] on differential (Hamilton-Jacobi-Isaacs) equations for Riemannian surfaces. Our approach using CAT(0) geometry is a significant departure, adapting well to a large variety of domains.

In a more general context, the progression from results about two-dimensional domains to domains of dimension three and higher is both challenging and vital in a number of problems across computational disciplines. These include optimal path-planning in robotics, ray tracing and visibility in computer graphics, and chattering phenomena in control theory. We do not find it a coincidence that in many of these related fields, those domains which serve as counterexamples to the simple "planar" models are those with an excess of positive curvature. See, for example, the paper of Canny and Reif [6], which demonstrates that finding shortest paths in a three dimensional PL Euclidean domain is NP hard, whereas in certain 3-d spaces which are CAT(0), it is of polynomial complexity [19]. The Canny-Reif construction secretly exploits positive curvature.

It is a theme forcefully demonstrated in the mathematics literature on CAT(0) geometry that a general property $\mathcal{X}$ which holds on both planar simply-connected domains and on higher dimensional convex sets should also hold on arbitrary CAT(0) domains.

What is more, the proof that property $\mathcal{X}$ holds should be the same as that of the planar case. This is a genuine hope against the *curse of dimensionality* which plagues many robotics motion planning problems. The present paper demonstrates a type of 'phase transition' for pursuit-evasion games in which the change is from *impossible* to *possible* as you move from general domains to those which are CAT(0).

More specific to the goals of this note, there are a number of future directions for inquiry. We have generalized many of the results on strategic pursuit from [22], [15]: many of the proofs lift directly from convex Euclidean to CAT(0) domains. We have not yet explored optimal evader strategies or analyzed optimal capture times (see, e.g., [22]). Based on the success of generalizing simple pursuit games to CAT(0) domains, an exploration of line-of-sight visibility games is in order. Likewise, a more physically realistic setting involving dynamic and/or kinematic constraints on the pursuer and evader pose interesting geometric questions: e.g., the case of dynamic constraints naturally leads to a discussion of curves with bounded curvature (cf. Dubins-type problems [8]). Such objects are likely amenable to techniques from CAT(0) geometry.

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References

[1] S. Alexander, I. Berg, R. Bishop, *Geometric curvature bounds in Riemannian manifolds with boundary*, Transactions Amer. Math. Soc., **339** (1993), 703-716.

[2] S. Alexander, R. Bishop, *The Fary-Milnor theorem in Hadamard spaces*, Proc. Amer. Math. Soc. 126 (1998), 3427–3436.

[3] A. Alexandrov, Y. Reshetnyak, General Theory of Irregular Curves, Kluwer Academic Publishers, Dodrecht, Boston, London, 1989.

[4] M. Bridson, A. Haefliger, Metric Spaces of Non-positive Curvature, Springer-Verlag, 1999.

[5] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry, Graduate Studies in Mathematics, Vol. 33, Amer. Math. Soc., Providence, RI, 2001.

[6] J. Canny and J. Reif. *Lower bounds for shortest path and related problems.* Proc. 28th Ann. IEEE Symp. Found. Comp. Sci., (1987) 49-60.

[7] B. Dekster, *The length of a curve in a space of curvature* $\leq K$, Proc. Amer. Math. Soc. 79 (1980), 271-278.

[8] L. Dubins. *On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents.* American Journal of Mathematics 79, (1957) 497-516.

[9] R. Ghrist, J. O'Kane, and S. LaValle, *Pareto optimal coordination on roadmaps*, Proc. Workshop Alg. Foundations of Robotics, 2004.

[10] L. Guibas, J.-C. Latombe, S. LaValle, D. Lin, and R. Motwani, *A visibility-based pursuit-evasion problem*, Inter. J. Comput. Geom. & Applications 9:4-5 (1999), 471-.

[11] N. Hovakimyan and A. Melikyan, *Geometry of pursuit-evasion on second order rotation surfaces*, Dynamics & Control 10(3) (2000), 297-312.

[12] R. Isaacs, Differential Games. Wiley Press, NY, 1965.

[13] V. Isler, S. Kannan, and S. Khanna, *Locating and capturing an evader in a polygonal environment*, in Proc. Workshop Alg. Foundations of Robotics, 2004.

[14] V. Isler, D. Sun. and S. Sastry, *Roadmap based pursuit-evasion and collision avoidance*, in Proc. Robotics, Systems, & Science, 2005.

[15] S. Kopparty and C. Ravishankar, *A framework for pursuit-evasion games in* $\mathbb{R}^n$, Information Proc. Letters, 96 (2005), 114-122.

[16] A. Kovshov, *The simple pursuit by a few objects on the multidimensional sphere*, Game Theory & Applications II, L. Petrosjan and V. Mazalov, eds., Nova Science Publ., (1996) 27-36.

[17] C. Maneesawarng and Y. Lenbury, *Total curvature and length estimate for curves in* $\mathrm{CAT}(K)$ *spaces*. Differential Geom. Appl. 19:2 (2003), 211–222.

[18] A. Melikyan, Generalized Characteristics of First Order PDEs, Birkhauser, 1998.

[19] J. Mitchell and M. Sharir, *New results on shortest paths in three dimensions*, Proc. 20th Annual ACM Symposium on Computational Geometry, (2004), 124-133.

[20] T. Parsons, *Pursuit evasion in a graph*, in Theory & Application of Graphs, Y. Alavi and D. Lick, eds., Springer Verlag, (1976), 426-441.

[21] Y. Reshetnyak, *Nonexpanding maps in a space of curvature no greater than* K, Sibirskii Mat. Zh. **9** (1968), 918-928 (Russian). English translation: *Inextensible mappings in a space of curvature no greater than* K, Siberian Math. Jour. **9** (1968), 683-689.

[22] J. Sgall, *Solution of David Gale's lion and man problem*, Theor. Comp. Sc. **259** (2001), 663-670

[23] I. Suzuki and M. Yamashita, *Searching for a mobile intruder in a polygonal region*. SIAM J. Comput. 21:5 (1992) 863-888.

[24] R. Vidalm O. Shakernia, H. Kim, D. Shim, and S. Sastry, *Probabilistic pursuit-evasion games: theory, implementation, and experimental evaluation*. IEEE Trans. Robotics & Aut. 18 (2002), 662-669.