Robotics: Science and Systems 2006
Philadelphia, PA, USA, August 16-19, 2006

Inverse Kinematics for a Serial Chain with Joints under Distance Constraints

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***Abstract*— Inverse kinematics (IK) problems are important in the study of robotics and have found applications in other fields such as structural biology. The conventional formulation of IK in terms of joint parameters amounts to solving a system of *nonlinear equations*, which is considered to be very hard for general chains, especially for those with many links.**

In this paper, we study IK for a serial chain with joints under distance constraints, in particular, either a spatial chain with spherical joints, or a planar chain with revolute joints (in this paper we ignore other constraints such as joint limits and link collision-free constraints, a common approach in studies of inverse kinematics). We present a new set of geometric parameters, which are not joint angles, for such chains, and use a novel approach to formulate the inverse kinematics as a system of *linear inequalities*, which is an exact, not an approximate, formulation of the IK problem. It follows that the IK problem for such a chain with an arbitrary number of joints can be done efficiently in many ways.

Under our new formulation, the set of solutions for an IK problem (as specified by the positions of the two end points of the last link), and more generally the set of solutions for all IK problems, is essentially *piecewise convex*. Our approach can also be generalized to other linkages such as those with prismatic joints sandwiched between rotational joints and with multiple loops that have a tree decomposition of triangles. The efficient algorithms and nice geometry entailed by piecewise convexity considerably simplify IK related problems, including motion planning, in the systems under study, and thus broaden the class of practical mechanisms at the disposal of robot designers.

I. Overview

Inverse kinematics (IK) is important in robotics and is indispensable in the design, analysis, planning and control of robot linkage systems such as robotic hands, limbs and humanoid robots. Given a serial chain and its end link configuration, the IK problem is to find joint parameters that achieve the specified end link configuration. This arises in many robotics problems, such as multi-linked manipulators and mobile robots, in which a linkage like a finger or leg needs to touch an object with a certain pose. It has recently been noted that some biological molecule structure problems [1], [2] can also be formulated as IK problems and thus benefit from IK approaches and results.

A. Prior approaches

Conventionally, IK problems are formulated with respect to joint parameters like linear displacements (for prismatic joints) or angles (for rotational joints). Such parameters correspond directly to the actuation of the joints, and thus become natural choices for linkage parameters. However, in general the end link configuration is a complicated function of joint parameters, especially when there are rotational joints in the chain, because the end effector configuration involves nonlinear functions of trigonometric terms of joint angles. Consequently, the IK problem of solving for the joint parameters in terms of a given end link configuration is considered to be very hard, especially for chains with many links.

IK has been intensively studied in robotics and mechanical design. Here we have no room to do justice to the large body of relevant literature; references [3]–[15] include books with chapters on IK, recent surveys, and papers containing representative techniques as well as recent formulations of IK problems using distance constraints and vector equations. In brief, prior work has provided two broad categories of solution methods for IK problems, numerical and analytical (or closed form). Numerical methods may generate all possible solutions or just one for an IK problem; they are applicable to general chains including chains with different numbers of links and different types of joints, but normally they are slow, require small step sizes in computation, and provide little insight on the solutions. Analytical methods have different traits: they provide insights on the solutions and are fast with the given analytical formula for the solutions, but generally new analytical solutions must be developed specifically for each type of linkage. Most, if not all, industrial robots use only mechanisms with known closed form IK solution [3]. It would be desirable to find a method that combines traits from each approach, one that is fast, can be applied to quite general linkages, and provides insight.

A good example of the state of the art is the theory of the $6R$ manipulator, a spatial chain of 6 links connected by 6 revolute joints. Using dialytical elimination, a method from algebraic geometry, researchers have proved one of the most important IK results: a $6R$ manipulator may have up to 16 IK solutions for a given end link configuration [9]–[11]. This leads to IK solutions for various related chains, such as those obtained by swapping out some revolute joints in $6R$ for prismatic joints. But these successes have depended on researchers' ingenious discovery of special structures in the system of polynomials that allow dialytical elimination.

IK for a serial chain is closely related to the study of closed chain systems. For an IK problem, since the end link configuration is specified we can supplement the real chain links with a virtual link from the base joint to the last link of the chain, thereby forming a closed chain in a loop configuration. In other words, the IK problem for a serial chain is essentially equivalent to the problem of generating closure configurations for a loop. Indeed, the intrinsic connection between these two problems has been recognized and explored. For example, the term "loop closure constraint" is used in both problems, and earlier methods developed for closed chains generate a loop configuration by breaking the loop into two serial sub-chains and then using random gradient descent [16] or a combination of forward and inverse kinematics [17] to try to make these two sub-chains meet. The efficiency of these approaches decreases rapidly for chains with many links. To address this issue, researchers have developed several closure configuration generation methods that can be adapted to solve the IK problem, such as the random loop generator [18] and iterative constraint relaxation [19], which have considerably improved performance over earlier methods.

In this paper, we study IK for a serial chain with rotational joints under distance constraints, in particular, a spatial chain with spherical joints or a planar chain with revolute joints. We ignore other constraints such as joint angle limits and the link collision-free constraint, a common approach in the study of inverse kinematics; also, for most of the paper we assume the distances between adjacent joints are fixed, to model rigid links, although we later briefly describe the generalization of our approach to variable distances, which can be used to model some prismatic joints. To simplify the discussion we will call a planar serial chain with n revolute joints a planar nR chain and a spatial serial chain with n spherical joints a spatial nS chain. Note that these linkages can also be used to model points under distance constraints in the plane and in space.

Prior work [20]–[23] on these systems, from the perspectives of closed chains and polygons, has described the topology of the set of closure configurations, which is essentially equivalent to the set of the solutions for an IK problem of a serial chain (also called the ***self-motion space***). In particular, it has been proved that the configuration space of a spatial loop with spherical joints, if not empty, has only one connected component, while the configuration space of a planar loop with revolute joints has either one or two connected components, depending on the number of "long links" (a technical term, cf. [22]). Armed with the knowledge of the global structure of the configuration space, researchers [20], [22] have also developed polynomial time complete planners that guarantee to generate a path between two closure configurations in the same connected component; the generated path satisfies the closure constraints but may involve link crossing (collision) and violation of joint angle limits. In their recent work, Trinkle and Milgram [22], [23] proved, using joint angle parameters, that the configuration space of a closed chain with generic link lengths is a manifold. More specifically, for a planar loop with n links having 3 long links, the closure configuration space is a pair of $(n-3)$-dimensional tori each coordinatized by

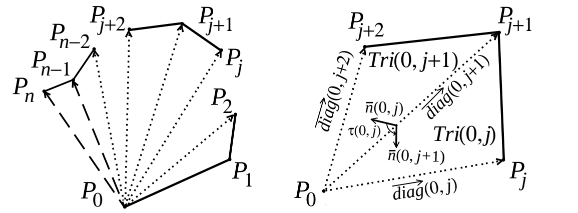


Fig. 1. Geometric Parameters: Diagonal Lengths and Dihedral Angles

the joint angles of the $n-3$ short links. Thus for this type of planar loop, given two closure configurations in the same connected component, linear interpolation of the $n-3$ joint angles (with the usual identification of angle 2π with 0) is a valid path: the closure configuration space for a planar chain with revolute joints and 3 long links, coordinatized by joint angles derived from the short links, is practically convex.

One problem not yet addressed in the aforementioned prior work on chains with joints under distance constraints is how to generate the closure configuration of a closed chain, or equivalently, how to solve IK for a serial chain. A closure configuration generation method [24] specifically designed for closed chains with spherical joints assumes that a closure configuration is given and then generates more closure configurations by selecting a subchain of the closed loop and then flipping (rotating) it along the line connecting the two end-joints of that subchain. This simple algorithm works for loops with arbitrarily many chains. But it needs at least one seed closure configuration to start the flipping process. The general closure configuration methods mentioned earlier can certainly be applied. However, these methods still have difficulty with chains of many links, say over 100; and, as usual with numerical methods, they provide little insight on the solution sets.

B. Our results

For our study of IK, we assume that the base joint is fixed and the positions of the two end points of the last link are given as the specification for an IK problem. In place of joint angles, we use another equally geometric set of parameters for the chain. We will show a novel reformulation of the loop closure constraint obtained by breaking a loop into an open chain of triangles and using triangle inequalities. Our parameters are not joint parameters; they consist of some inter-joint distances and some triangle orientation data. With our parameters, the loop closure constraint can be formulated, exactly not approximately, as a set of *linear inequalities*. It will follow that the IK problem for an arbitrary number of joints reduces to solving linear inequalities, which can be efficiently done in many ways including linear programming.

Specifically, let ${}^kIK$ be the set of solutions for an IK problem of a given serial chain with rotational joints under distance constraints and n rigid links of specified lengths in space ($k = 3$) or the plane ($k = 2$). We can choose an arbitrary joint of the chain to be an *anchor*; in this paper, we use the base joint as the anchor. In general we will call an object "anchored" if it includes the anchor joint. Draw the anchored diagonals from the anchor to other joints. As shown in Fig. 1, these diagonals, along with the links, partition the loop into an open chain of triangles, with the configuration of the last triangle known for each given IK problem. In a sense, our new parameters are these anchored triangles themselves. We will extract more conventional parameters from these triangles including diagonal lengths $\bar{\mathbf{r}}(0)$ and triangle orientation parameters (dihedral angles for spatial nS and planar nR chains, or orientation signs for planar nR chains) in section II. We will show that our new parameters (anchored diagonal lengths, anchored triangle orientation parameters) are indeed coordinates for the set of all IK solutions not having any *degenerate* anchored triangles. Note that a triangle is degenerate when its three vertices are collinear. We will briefly describe how to generalize our new parameters to handle singular IK solutions.[1]

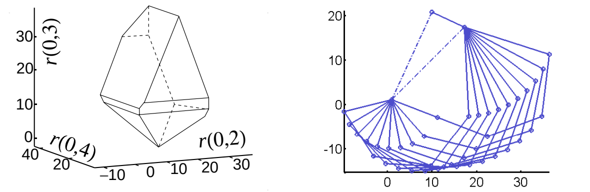
We will explain in section III that the anchored triangle orientation parameters are independent of the loop closure constraints. We will formulate the loop closure constraints as a set of linear inequalities in diagonal lengths $\bar{\mathbf{r}}(0)$ and show that the set of all diagonal lengths feasible for the solutions of a given IK problem is a readily computable convex polyhedron $IKStretch(0)$ that is non-empty if and only if ${}^kIK$ is. Using our new parameters and the new formulation of the loop closure constraints, with proper treatments of singular configurations, the ${}^2IK$ of a planar nR chain is essentially tiled by 2^{n-2} copies of the convex polyhedron $IKStretch(0)$ joined into one connected component or two via proper boundary identification, making it piecewise convex in a very tractable sense; and the ${}^3IK$ of a spatial nS chain is essentially the product of a high-dimensional torus and the convex polyhedron $IKStretch(0)$, making it what we call "practically convex" since a torus is easily cut open into a cube, which is convex. (We use the word "essentially" here because a certain lower-dimensional subset of IK solutions, namely that of the "super-singular" ones, to be defined later, does need some extra care.) To simplify our description, we will say that both ${}^3IK$ and ${}^2IK$ parameterized by our new parameters are piecewise convex, bearing in mind that the ${}^3IK$ has only one piece and involves a torus factor space.

Although we cannot give details here, it will become clear that our parameters and the linear inequality formulation of the loop closure constraint can easily be generalized to a chain with prismatic joints sandwiched between rotational joints under distance constraints, and to some linkage systems involving multiple loops. Furthermore, the set of all IK solutions, which is the union of the solution sets for all IK problems, for the systems under study is still piecewise convex.

C. Discussion

A few examples will highlight the connection between our work and prior results, as well as the new insights our work

[1]A configuration that is singular in the traditional sense but includes no degenerate *anchored* triangle poses no problems for our new parametrization, so for our purposes it is non-singular. We will discuss the role of anchor choice in future papers.



(a) IKStretch for a planar $6R$ chain or a spatial $6S$ chain (b) Some IK solutions for a planar $6R$ chain

Fig. 2. IK examples for a 6-bar chain with link lengths $(11, 12, 13, 14, 20, 8)$ and distances 23 and 21.6 from base joint to last joint and tip of the chain. The dashed-dot lines in (b) and in later figures are the diagonals from the base joint to the endpoints of the last link, which are known and fixed for each IK problem.

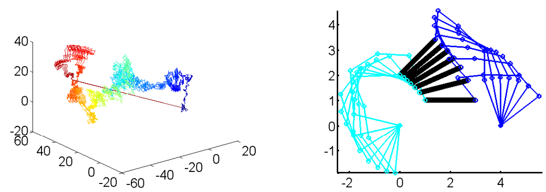
provides. The IK problem for a serial chain with 3 links has well known solutions: the IK problem, if not infeasible, has two solutions for a planar $3R$ chain (one elbow-up, the other elbow-down), and a circle of solutions (free rotations around a line of the joint with unspecified position) for a spatial $3S$ chain. Our new approach recovers these results.

For a planar $6R$ chain with generic link lengths, our approach constructs the IK solution set from 16 copies of a 3-dimensional convex polyhedron.[2] Fig. 2(a) shows the 3-dimensional convex polyhedron $IKStretch(0)$ for the 6-bar IK problem specified in the caption. Fig. 2(b) shows some IK solutions for a planar $6R$ chain, with the base joint at the origin and the two joints of the end link at the positions specified for some IK problem.

Our linear inequality formulation of IK problems for a chain with rotational joints under distance constraints allows efficient computation of IK solutions for chains of this class of arbitrarily many joints. In addition to linear programming, we have developed a suite of methods [25] for solving IK problems, some outperforming linear programming. For example, one of our best generators, which we call the diagonal sweeping method, can generate one set of diagonal lengths for an IK problem for a loop of 1000 links in 19 milliseconds in Matlab on a desktop computer. This one set of diagonal lengths can be used to generate 2^{998} related IK solutions for a planar chain, or be combined with a 998-dimensional torus, $(S^1)^{998}$, to generate a family of related IK solutions for a spatial chain. Fig. 3(a) shows some IK solutions for a spatial $1000S$ chain.

Our new formulation and results for IK problems of chains with rotational joints under distance constraints have profound implications for such systems. As is well known in robotics, system designers gain important insight from knowledge of IK solutions and their dependence on the underlying parameters. Partly due to the importance of IK problems in robotics and the lack of general solving strategies, mechanisms used in the design of robots have generally tended to be those known to have analytical IK solutions. The nice geometry of the IK solutions of planar nR and spatial nS chains, and the resulting efficient algorithms, will provide robot designers with

[2]Note that the class of systems studied here does not contain what is commonly called a $6R$ chain, in which 6 links are connected by revolute joints in space rather than in the plane.



(a) Some IK solutions for a spatial $1000S$ chain

(b) Manipulation path of two fingers, each being a $5S$ chain

Fig. 3. IK related examples for spatial nS chains

a broader class of practical mechanisms.

In particular, the piecewise convexity of an IK solution set and, more strikingly, the piecewise convexity of the set of all IK solutions, considerably simplify the kinematics problems of the linkage system such as path planning and manipulation planning, which is a great benefit on its own and further makes the system more appealing for robot design. Essentially, for a spatial nS chain and two points on its self-motion manifold, linear interpolation between our parameter values of the two points stays on the manifold; this need not be the case with the joint angle parameters. Fig. 3(a) actually shows a linear path between two solutions (with different diagonal lengths and triangle orientations) for one $1000S$ chain IK problem.

Fig. 3(b) shows a manipulation path of a block grasped by two fingers (of different colors), each a spatial $5S$ chain. Even with the fingertips changing their positions during the manipulation process, we can still use the linear interpolation of the new linkage parameters and the object configurations of the two given configurations to find a path for the system, since the set of all IK solutions is piecewise practically convex. Path planning for planar nR chains is a little more involved than for spatial nS chains, since their configuration spaces are more complicated. Fig. 2(b) actually pictures a linear path between two IK solutions in a single copy of $\mathit{IKStretch}(0)$; the validity of this path is guaranteed by the convexity of $\mathit{IKStretch}(0)$. To join two IK solutions not on the same copy of $\mathit{IKStretch}(0)$ of ${}^2IK$ for a planar nR chain, we need to find critical intermediate singular configurations that can connect the start and goal copies possibly via some other copies; again, efficient algorithms like linear programming and diagonal sweeping can be used to compute these critical configurations. Thus, as shown in our paper [26] motion planning for planar nR chains can also be efficiently done.

II. New Parameters

A. Problem formulation

Consider a serial chain consisting of n rigid links with link lengths $l_j > 0, j = 0, \ldots, n-1$, connected by spherical joints (for a chain in three-dimensional space) or revolute joints (for a chain in the plane). Denote the end points of the links in the chain, including the joints and the tip of the last link, by $P_j, j = 0, \ldots, n$. With a slight abuse of notation, we will also write P_j for the coordinate of the point P_j.

For this paper, we assume that a chain configuration is completely specified by its joint positions. In such a system model, each link is represented by two points at its ends. For a general rigid link, two points may not be enough to specify the link configuration—for instance, if the link is not symmetric about the line passing through its endpoints. Nevertheless, we adopt this formulation since it still captures the essence and complexity of the IK problem.

For the IK problem, a reference frame is given and the configuration of the first joint and the last link is specified; the problem is to find joint variables achieving the required end link configuration. Conventionally, a local frame of the last link needs to be defined and its relative position and orientation with respect to the reference frame, which can be described by a rigid body transformation matrix in $SE(k)$, are used to describe the last link configuration. In this paper, instead of directly using a transformation matrix, we use the positions of the last two points P_n and P_{n-1} as the specified values for a given IK problem. These two values should be easily derived from the local frame of the last link, since a conventional local frame for the link will have the origin at one endpoint and one of the axis directions as the link direction. In the rest of this paper, we will refer to the first joint P_0 and the last two endpoints P_{n-1}, P_n as the end joints of our chain.

A spherical joint i can be parameterized by two spherical angles, say α_i and β_i, with $\alpha_i \in (-\pi, \pi]$, $\beta_i \in (-\pi/2, \pi/2]$. A revolute joint can be parameterized by α_i, or equivalently, restricting β_i used in a spherical joint to zero. With our notation, the conventional formulation of an IK problem respect to joint angles can be described as follows.

$$IK : (P_0, P_{n-1}, P_n, \bar{l}) \to (\alpha_i, \beta_i, i = 0, \ldots, n-1) \qquad (1)$$

Here $\bar{l} = (l_0, \ldots, l_{n-1})$ is the n-tuple of link lengths. Clearly, the joint angles of the last link can easily be computed from its two endpoint positions; the main difficulty of the inverse kinematics lies in computing the joint angles of the subchain from link 0 to link $n-2$.

For the development of our parameters, we note that the IK problem can also be formulated in terms of joint positions. This is because with all joint positions computed, it is straightforward to obtain joint angles.

$$IK : \ (P_0, P_{n-1}, P_n, \bar{l}) \to (P_i, i = 1, \ldots, n-2) \qquad (2)$$

Conversely, as shown in this section, a given set of our parameters taken together with the given end joint positions can be used to uniquely determine all joint positions. In other words, the following function is well defined for our parameters, temporarily denoted $Param$.

$$f : (P_0, P_{n-1}, P_n, \bar{l}, Param) \to (P_i, i = 1, \ldots, n-2).$$

Section III addresses how to solve inverse kinematics problems with respect to our parameters,

$$IK : (P_0, P_{n-1}, P_n, \bar{l}) \to Param.$$

B. Serial spatial chains with spherical joints: new parameters

For a given serial chain with a fixed base and n links, use the base (joint 0) as the anchor joint and draw the diagonals from the anchor to other joints $P_j, j = 1, 2, \ldots, n$, as shown in Fig. 1. Note that genuinely "diagonal" vectors correspond to $j = 2, \ldots, n$; for $j = 1$, we get link 0 that is incident on the anchor joint. It will become clear soon that treating this link also as a diagonal simplifies our description. The diagonals and chain links define $n-1$ triangles all sharing the anchor joint; a configuration of the chain is called *singular* for that anchor in case one or more of these triangles is degenerate, *i.e.*, reduces to a line segment. Denote the vector from the anchor joint 0 to joint j by $\overrightarrow{diag}(0, j)$ ($j = 1, \ldots, n$), its length by $r(0, j)$, and its corresponding unit directional vector by $\overline{diag}(0, j)$. Of course $\overline{diag}(0, j)$ is not defined in case $r(0, j) = 0$, but this happens if and only if joint $j \in \{2, \ldots, n\}$ has become coincident with the anchor, which is a special type of singular configuration that we call *super-singular*. In this paper, we focus on non-singular IK solutions, which form an open dense subset of the whole IK solution set.

Each pair of adjacent non-degenerate triangles defines an angle, called a dihedral angle, that reflects the relative orientation of the triangle pair. Denote the triangle formed by the anchor joint and consecutive endpoints j and $j+1$ by $Tri(0, j)$ ($j = 1, \ldots, n-1$), and its unit normal by

$$\overline{n}(0, j) = normalize(\overrightarrow{diag}(0, j) \times \overrightarrow{diag}(0, j+1))$$

(where $\times$ is vector cross product). Define the dihedral angle between consecutive triangles $Tri(0, j)$ and $Tri(0, j+1)$ to be the angle for rotating $\overline{n}(0, j)$ to $\overline{n}(0, j+1)$ about their shared diagonal $\overline{diag}(0, j+1)$, and denote it $\tau(0, j)$. Thus

$$\overline{n}(0, j+1) = Rot(\overline{diag}(0, j+1), \tau(0, j))\overline{n}(0, j), \quad j = 1, \ldots, n-2 \tag{3}$$

where $Rot(\overline{diag}(0, j+1), \tau(0, j))$ is the matrix representing rotation about the unit diagonal vector $\overline{diag}(0, j+1)$ by the angle $\tau(0, j)$.

For the IK problem, the last two diagonal lengths $r(0, n-1)$ and $r(0, n)$, and the orientation of the last triangle, can be computed from the given endpoint position P_0, P_{n-1}, P_n. So there are $n-3$ unknown diagonal lengths and $n-2$ unknown dihedral angles, which we can group together into column vectors. Our new parameters for the IK problem (defined with respect to the anchor 0) are $(\bar{\mathbf{r}}(0), \overline{\tau}(0))$, where $\bar{\mathbf{r}}(0)$ is the set of diagonal lengths (well defined everywhere on ${}^kIK$), and $\overline{\tau}(0)$ is the set of dihedral angles (well defined on non-singular configurations).

$$\begin{aligned} \bar{\mathbf{r}}(0) &= [r(0, 2), \ldots, r(0, n-2)]' \\ \overline{\tau}(0) &= [\tau(0, 1), \ldots, \tau(0, n-2)]' \end{aligned} \tag{4}$$

As stated earlier, we will show in the next section how to compute these parameters for a given set of end joint positions. Here we assume that we have a set of diagonal lengths and dihedral angles for a given set of end joint positions, and we will show how to compute other joint positions.

$$f\colon (P_0, P_{n-1}, P_n, \bar{l}, \bar{\mathbf{r}}(0), \overline{\tau}(0)) \to (P_j, j = 1, \ldots, n-2) \tag{5}$$

Input: $P_0, P_{n-1}, P_n, \bar{l}, \bar{\mathbf{r}}(0), \overline{\tau}(0)$
Output: $P_j, j = 1, \ldots, n-2$
Algorithm:
1. $\overline{diag}(0, n) = normalize(P_n - P_0)$
2. $\overline{diag}(0, n-1) = normalize(P_{n-1} - P_0)$
3. $\overline{n}(0, n-1) = normalize(\overline{diag}(0, n-1) \times \overline{diag}(0, n))$
4. for $j = n-2$ to 1
5. $\quad \overline{n}(0, j) = Rot(\overline{diag}(0, j+1), -\tau(0, j))\overline{n}(0, j+1)$
6. $\quad \gamma(0, j) = \text{acos}\left(\dfrac{r^2(0, j) + r^2(0, j+1) - l_j^2}{2.0 * r(0, j) * r(0, j+1)}\right)$
7. $\quad \overline{diag}(0, j) = Rot(\overline{n}(0, j), -\gamma(0, j))\overline{diag}(0, j+1)$
8. $\quad P_j = P_0 + r(0, j)\overline{diag}(0, j)$
9. endfor

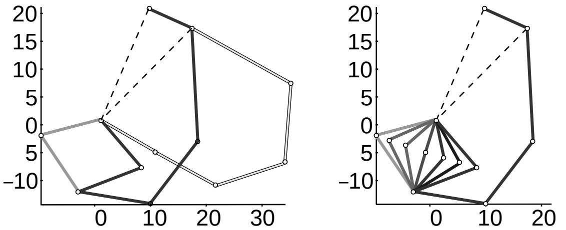
Fig. 4. Algorithm for Computing Joint Positions of a Spatial Chain

Our algorithm computes the joint coordinates incrementally and is given in Fig. 4. The first three lines in the algorithm compute the diagonal and normal vectors of the last triangle. Then the loop between lines four and nine computes the joint positions, starting at the one closest to the last triangle (which is joint $n-2$) and moving down to joint 1, with each execution of a loop determining one joint position. Inside the loop, line 5 computes the normal of a current triangle, which is adjacent to the most recently determined triangle. Note that the current triangle shares a diagonal, and thus two joints, with its known neighbor, and thus has only one unknown joint position and one unknown diagonal vector. Line 6 uses the law of cosines to determine the angle between the two diagonals of the current triangle. Line 7 computes the direction of the unknown diagonal vector of the current triangle, and line 8 uses the computed diagonal direction and given diagonal length to compute the joint position.

The algorithm must be modified to handle singular configurations. If a configuration is singular but not super-singular—that is, if at least one triangle is degenerate but no diagonal has length 0—then the modification is simple. Although there is no natural way to select any particular unit vector to play the role of $\overline{n}(0, j)$ at the corresponding "singular step" in our algorithm, any unit vector in the well-defined plane perpendicular to all sides of the degenerate triangle can serve as $\overline{n}(0, j)$. In practice, we can make a definite choice by setting $\overline{n}(0, j) = \overline{n}(0, j+1)$, and choosing $\overline{n}(0, n-1)$ at random in case $Tri(0, n-1)$ is degenerate. Super-singular configurations have even greater indeterminacy and require some other parameters, which will be described in our future papers. Singular and super-singular configurations play an important role in using our parameters to reconstruct the global topological structure of the configuration space, especially for planar nR chains as shown in paper [26].

C. Serial planar chains with revolute joints: new parameters

For a planar chain, we define the same set of anchored diagonals and triangles as for a spatial chain, and use the anchored diagonal lengths $\bar{\mathbf{r}}(0)$ as part of the parameters. Since all triangles of a planar chain are in one plane, each dihedral angle—if well defined—can be only 0 or π, depending on



(a) Planar $6R$ chain: via a singular configuration (light gray)

(b) Spatial $6S$ chain : via flipping

Fig. 5. Changing one triangle orientation (black to dark gray)

whether the two triangles between which the angle lies have the same or opposite orientations. Therefore, a straightforward adaptation of the spatial chain parameters to planar chains is to use the same set of parameters $\overline{\mathbf{r}}(0), \overline{\tau}(0)$, taking note that each dihedral angle is either 0 or π. An alternative representation of triangle orientations is by direction signs of triangle normals. If we consider our planar chain to lie in the XOY plane in space, then all points of the chain have zero (0) z coordinates, and all normals of the non-singular triangles are in the z direction, either positive or negative. Therefore, we can use the positive $(+)$ and negative $(-)$ signs to represent the orientations of these triangles. We will define the orientation sign of a singular anchored triangle to be 0.

The discrete dihedral angle values and orientation signs are equivalent parameters for the anchored triangle orientations; and each set can be changed to the other easily. However, they have different characteristics and may be suitable for different types of problems. In particular, the orientation sign parameters are particularly convenient for determining if any two given configurations have opposite orientations in any of their anchored triangles. In the plane, change of triangle orientations (such as the two drawn in in Fig. 5(a), which differ in only one triangle orientation, black versus dark gray) must go through singular configurations of the triangle (like the one in light gray—see [26] for more details). But in space, changing triangle orientation is much easier since we can just flip that triangle (Fig. 5(b)). Henceforth, we use orientation signs in this paper for planar chain triangle orientations.

Denote the orientation sign of $Tri(0,j)$ by $s(0,j)$. For the IK problem, the orientation of the last triangle can be computed from the specified end joint positions. The orientation signs of all unknown triangles will be grouped into a column vector and referred to as $\overline{\mathbf{s}}(0)$ in the rest of the paper.

$$\overline{\mathbf{s}}(0) = [s(0,1) \ \cdots \ s(0,n-2)]' \tag{6}$$

$$s(0,j) \in \{+,-,0\}, \ j = 1,\ldots,n-2 \tag{7}$$

The planar counterpart of algorithm 4 for computing joint positions from end joint positions and our parameters uses ideas similar to those used for the spatial case. It is considerably simpler and is not included here due to space limit.

III. The Inverse Kinematics solutions

In this section we describe how to compute IK solutions in terms of our new parameters. We now indicate the index of the anchor, 0, explicitly by writing ${}^3IK(0)$ for the set of all IK solutions for a spatial chain and ${}^2IK(0)$ for a planar chain, parameterized by our new parameters anchored at the base joint. Our task is to compute the possibly empty sets

$$\begin{aligned} {}^3\widetilde{IK}(0) = \{(\overline{\mathbf{r}}(0), \overline{\tau}(0)) \mid \ & \overline{\mathbf{r}}(0) \text{ and } \overline{\tau}(0) \text{ reach the} \\ & \text{specified end joint position}\}, \end{aligned} \tag{8}$$

$$\begin{aligned} {}^2\widetilde{IK}(0) = \{(\overline{\mathbf{r}}(0), \overline{\mathbf{s}}(0)) \mid \ & \overline{\mathbf{r}}(0) \text{ and } \overline{\mathbf{s}}(0) \text{ reach the} \\ & \text{specified end joint position}\} \end{aligned} \tag{9}$$

attained by our parameter values on the set of solutions where they can be defined.

Note that by drawing a diagonal from the base joint to the end joint, we define a virtually closed chain such that solving the IK problem is equivalent to generating chain parameters that can keep the chain closed and place the end points at the specified positions. In other words, we need to find the set of chain parameters that satisfy the *loop closure constraints*. We further note that by drawing the diagonals from the base joint to all other joints, we *decompose the closed chain into a chain of triangles* with the last triangle completely known from the inverse kinematics specification.

A. The sets of parameters $\overline{\mathbf{r}}(0)$ and $\overline{\tau}(0)$/$\overline{\mathbf{s}}(0)$ are uncoupled

Our first observation is that the triangle orientations are independent of the loop closure constraints. In fact, given a spatial loop configuration, changing one dihedral angle is equivalent to flipping part of the serial chain about the corresponding diagonal vector while keeping the other part fixed, as shown in Fig. 5(b). Similarly, changing the orientation sign of a non-singular anchored triangle amounts to flipping one side of the triangle to the other, as shown in Fig. 5(a). Clearly flipping one or more anchored triangles can lead to different configurations, and yet maintains the link lengths, the anchored diagonal lengths and the loop closure.

Therefore, the diagonal lengths and the triangle orientation parameters (dihedral angles or orientation signs as applicable) are uncoupled; and the loop closure constraints pose no restrictions on feasible values of the anchored triangle orientations. More specifically, for a spatial nS chain, each of its anchored dihedral angles can be any value in $[-\pi,\pi]$, which can be identified with a flat circle obtained by identifying the endpoints of an interval of length 2π; for a planar nR chain, each of the orientation signs for its non-singular anchored triangles can indeed be any value in $\{+,-\}$.

B. The set of all feasible diagonal length values

Now that we have uncoupled the diagonal length parameters from the triangle orientation parameters, we formulate the closure constraints on the diagonal lengths. As described earlier, by drawing diagonals from the base joint to the joints not adjacent to it, we define $n-1$ triangles, with links and diagonals as triangle sides. The last triangle is completely known for any given case of IK problems. It is conceivable that an inappropriate value of a diagonal length may make the diagonal too long or too short to form a triangle and thus may not be part of the IK solution. **Our second observation** is that for a given set of diagonal lengths, the loop closure constraint can be satisfied if and only if they can form all $n-2$ unknown triangles with the links. From basic plane geometry, we know that a set of three non-negative numbers $\{l1, l2, l3\}$ can serve as the side lengths of a (possibly degenerate) triangle if and only if no side length is strictly greater than the sum of the other two side lengths. In other words, $\{l1, l2, l3\}$ must satisfy

$$l1 \leq l2 + l3,\ l2 \leq l3 + l1,\ l3 \leq l1 + l2 \qquad (10)$$

Furthermore, the triangle is non-degenerate if and only if all three inequalities are strict. Using these triangle inequalities, we can explicitly write out the loop constraints in terms of the lengths of the links and diagonals in the compact matrix format

$$T\bar{\mathbf{r}}(0) \leq \bar{\mathbf{b}}(0), \qquad (11)$$

where $\bar{\mathbf{r}}(0)$ is the vector of diagonal lengths, $\bar{\mathbf{b}}(0)$ is the vector of terms on the right hand side, and the $(3n-8) \times (n-3)$ matrix T has one row for each inequality in the following system.

$$\begin{array}{rcl} r(0,2) & \leq & l_0 + l_1 \\ -r(0,2) & \leq & -|l_0 - l_1| \\ r(0,j) - r(0,j+1) & \leq & l_j \\ -r(0,j) + r(0,j+1) & \leq & l_j \\ -r(j) - r(j+1) & \leq & -\ l_j \\ r(0,n-2) & \leq & l_{n-2} + |P_0P_{n-1}| \\ -r(0,n-2) & \leq & -\big|l_{n-2} - |P_0P_{n-1}|\big| \end{array} \quad \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} 2 \leq j \leq n-3 \qquad (12)$$

The first and last two lines in (12) are inequalities for the first and last unknown triangles, $Tri(0,1)$ and $Tri(0,n-2)$, and $|P_0P_{n-1}|$ is the length of the virtual link from P_0 to P_{n-1}. Each of these two triangles has two links and one diagonal as its sides. The middle three lines in (12) are for intermediate triangles $Tri(0,j)$ ($j = 2, \ldots, n-3$), each having one link and two diagonals as its sides.

Denote the side of all feasible $\bar{\mathbf{r}}(0)$ values to be $IKStretch(0)$, we have

$$IKStretch(0) = \{\bar{\mathbf{r}}(0) \mid T\bar{\mathbf{r}}(0) \leq \bar{\mathbf{b}}(0)\}. \qquad (13)$$

Note that all constraints on the diagonal vectors are *linear inequalities* and each one defines a closed half space. Thus $IKStretch_0$ is the intersection of half-spaces; by convexity theory, it is a convex polytope. Moreover, it is easy to see that, for a given set of positive link lengths, the lengths of all diagonal vectors are bounded between zero and the sum of all link lengths, which are all encoded in the triangle inequality constraints. So—again by convexity theory—we conclude that $IKStretch_0$ is a *convex polyhedron*, possibly empty, of dimension $n-3$ (for generic link lengths) or less.

C. The convexity of ${}^kIK$

Note that any point $\bar{\mathbf{r}}(0)$ in $\mathrm{Int}(IKStretch(0))$, the interior of $IKStretch(0)$, satisfies all triangle inequalities in (12) as strict inequalities without achieve equality in any of them. This means that none of the anchored triangles can be singular for the given diagonal lengths. In fact, $\mathrm{Int}(IKStretch(0))$ is the set of all $\bar{\mathbf{r}}(0)'s$ leading to non-singular IK solutions. Taking our two observations together, we can write the set of all non-singular IK solutions as the product of the two factor spaces as follows.

$${}^k\widetilde{IKNS}(0) = \mathrm{Int}(IKStretch(0)) \times {}^kIKFlip(0) \qquad (14)$$

where ${}^kIKFlip(0)$ is the set of feasible triangle orientation parameter values, with ${}^3IKFlip(0) = [-\pi, \pi]^{n-2}$ and ${}^2IKFlip(0) = \{+, -\}^{n-2}$.

${}^3IKFlip(0)$ is a high-dimensional torus, which can be easily cut open into a cube and thus can be viewed as practically convex. Therefore, ${}^3\widetilde{IKNS}(0)$ is practically convex. We describe this situation succinctly by saying that ${}^3IK$ is *practically convex*: our parameters give a one-to-one correspondence between its open dense subset of non-singular configurations and the convex set $\mathrm{Int}(IKStretch_0) \times [-\pi, \pi]^{n-2}$. Of course ${}^2IKFlip(0)$ is not convex (except for the trivial case $n = 2$), but the product $\mathrm{Int}(IKStretch(0)) \times {}^2IKFlip(0) = {}^2\widetilde{IKNS}(0)$ is the union of 2^{n-2} pairwise disjoint copies of the convex polyhedron $\mathrm{Int}(IKStretch(0))$, and our parameters identify the open, dense, generally disconnected set of non-singular configurations in ${}^2IK$. We say that ${}^2IK$ is *piecewise convex*.[3] We summarize our results in the following theorems.

Theorem *1:* Given a spatial closed chain of n rigid links connected by spherical joints and the positions of its base point P_0 and the endpoints of the last link P_{n-1}, P_n, the set ${}^3IK(0)$ of IK solutions with respect to the parameter set of diagonal lengths and dihedral angles, if not empty, is practically convex.

Theorem *2:* Given a planar closed chain of n rigid links connected by revolute joints and the positions of its base point P_0 and the endpoints of the last link P_{n-1}, P_n, the set ${}^2IK(0)$ of IK solutions with respect to the parameter set of diagonal lengths and triangle orientation signs, if not empty, is piecewise convex.

While we have no space to give details in this paper, we can prove that the anchor and the corresponding parameters determine a *stratification* of ${}^kIK$ in a sense familiar from topology and algebraic geometry—that is, a decomposition of ${}^kIK$ into finitely many smooth connected pairwise disjoint manifolds, its *strata*, of various dimensions (see, *e.g.*, [27]). Refer to paper [26] for the stratification of the self-motion space of a planar nR loop.

IV. Generalizations of our approach

Our parameters can be generalized to other type of chains and IK problems. For example, all link lengths and the end-to-end distance ($|P_0P_{n-1}|$) in our formulation of the inequalities (12) are assumed to be fixed. If we allow some link length, say l_k, to lie in an interval $[\underline{l}_k, \bar{l}_k]$, we can augment our parameters with this link length variable and introduce

[3] We will describe the strata of ${}^kIK$ in all dimensions in future papers, using parameters like those described here but somewhat more complicated, particularly for super-singular strata. Then, guided by the face structure of $IKStretch$, we reconstruct the global topology of ${}^kIK$ from its strata.

the additional constraint $\underline{l}_k \leq l_k \leq \bar{l}_k$, which is still linear and keeps the augmented $IKStretch$ convex. The changing length of a link can be used to model a prismatic joint.

For a chain with rotatable joints and a fixed base, we know that the workspace, *i.e.* the set of reachable positions, of its tip (and any point on the chain) is a spherical shell in general, which means that the end-to-end distance $|P_0P_n|$ has a range. So we can use a similar idea to augment our parameters with those for the last anchored triangle along with a range constraint on $|P_0P_n|$. Then the resulting augmented $IKStretch$ and IK include the solutions for *all* IK problems of the chain. Note that *the augmented $IKStretch$ is still a convex polyhedron*, and *the augmented set of the solutions of all IK problems still has convexity properties similar to those of the solution set of a single IK problem.*

Our approach can also be generalized to more complicated kinematic structures such as those involving multiple loops. Our decomposition of one loop into a serial chain of triangles, which allows us to decouple the parameters and to formulate the loop closure constraint as a set of triangle inequality constraints, generalizes to multiple loops that can be decomposed into a tree of triangles, carrying with it our parameters as presented in this paper. More substantial work will be needed for loops without a tree decomposition and with other joint types, as well as to incorporate other constraints such as joint limits and collision avoidance.

V. Summary

Inverse kinematics is a fundamental problem in robotics. The conventional formulation of IK in terms of joint parameters amounts to solving a set of nonlinear equations, a problem for which there is no general analytical solution. Partly due to the importance of IK problems in robotics and the lack of general solving strategies, designs for new robots tend to be limited to developments and combinations of mechanisms with known analytical IK solutions.

In this paper, we present a new set of geometric parameters, namely anchored diagonal lengths and triangle orientations, for solving the inverse kinematics of a serial chain with spherical joints in space or revolute joints in the plane. Formulated in our parameters, IK is a set of linear inequalities and can be efficiently solved in many ways [25] such as linear programming and diagonal sweeping. We also show that for a serial chain with rotational joints under distance constraints the solution set of an IK problem, if not empty, is practically convex. Indeed it is a pleasant surprise that the inverse kinematics for a chain with all rotational joints like a planar nR chain and a spatial nS chain can be formulated as a set of linear inequalities. As briefly outlined in the paper, our approaches can be generalized to other linkage systems, which broadens the class of practical mechanisms at the disposal of robot designers.

Acknowledgment

The authors thank the anonymous reviewers for their helpful comments.

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